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On Best Simultaneous Approximation in Banach Spaces

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Diaz and McLaughin [2, 3] and Dunham [4] have considered the problem of simultaneously approximating two continuous functions f_1 and f_2 by elements of C, a nonempty family of real-valued continuous functions on $[a, b]$. These results in a general setting have been given by Holland *et al.* [5] and Bosznay [1]. It is the aim of this paper to prove some further results under relaxed conditions.

Let C be a subset of a normed linear space X. Given any bounded subset F in X, define

$$d(F, C) = \inf_{c \in C} \sup_{f \in F} \|f - c\|.$$

An element $\mathbf{c}$ in C is said to be a best simultaneous approximation to set F if

$$d(F, C) = \sup_{f \in F} \|f - \mathbf{c}\|.$$

The following main theorem is given in [5].

Let X be a uniformly convex Banach space and let A be a closed bounded and convex subset of X. For any compact subset F of X, there exists a unique best simultaneous approximation to F from elements of A.

We prove the following theorem where the condition of uniform convexity has been relaxed.

THEOREM 1. *Let X be a strictly convex, Banach space, and C a weakly compact, convex subset of X. Then there exists a unique best simultaneous approximation from the elements of C to any given compact subset F of X.*

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Proof. By definition $\mathbf{c}$ in C is said to be a best simultaneous approximation to F if

$$d(F, C) = \sup_{f \in F} \|f - \mathbf{c}\|.$$

It is proved in [5] that the function ϕ defined by

$$\phi(c) = \sup_{f \in F} \|f - c\|$$

is convex and continuous and is, therefore, a weakly lower semicontinuous function on C. Since C is a weakly compact subset of X, ϕ attains its infimum at $\mathbf{c}$ in C, say. Therefore,

$$d(F, C) = \sup_{f \in F} \|f - \mathbf{c}\|.$$

Now we show that the best simultaneous approximation is unique.

Let c_1 and c_2, $c_1 \neq c_2$, be two best simultaneous approximations by elements of F, i.e.,

$$\sup_{f \in F} \|f - c_1\| = \sup_{f \in F} \|f - c_2\| = d, \qquad \text{say.}$$

Then $\sup_{f \in F} \|f - (c_1 + c_2)/2\| = d$ (see [5]).

Since F is compact there exists a $\mathbf{f} \in F$ with

$$\sup \left\| f - \frac{c_1 + c_2}{2} \right\| = \left\| \mathbf{f} - \frac{c_1 + c_2}{2} \right\| = d,$$

it then follows that

$$\|\mathbf{f} - c_1\| = d \qquad \text{and} \qquad \|\mathbf{f} - c_2\| = d.$$

Since X is strictly convex we get that $c_1 = c_2$.

The following theorem, due to Holland *et al.* [5], can be obtained as a corollary.

Let C be a closed, bounded and convex subset of a uniformly convex Banach spae X. Then for any compact subset F of X there exists a unique best simultaneous approximation to F from the elements of C.

Since a uniformly convex Banach space is strictly convex and reflexive. Also, C is weakly compact so the result follows from Theorem 1.

In [5] the following interesting theorem has been proved.

Let C be a finite dimensional subspace of a strictly convex normed linear space X. Then there exists one and only one best simultaneous approximation from the elements of C to any given compact subset F of X.

A natural question is suggested by this theorem. Is the hypothesis of finite dimension really necessary?

We prove the following

THEOREM 2. *Let X be a strictly convex normed linear space and C a reflexive subspace of X. Then for any nonempty compact subset F of X there exists one and only one best simultaneous approximation in C.*

Proof. By definition

$$d(F, C) = \inf_{c \in C} \sup_{f \in F} \|f - c\|.$$

The function $\phi: C \to R$ defined by

$$\phi(c) = \sup_{f \in F} \|f - c\|$$

is continuous and convex on C [5]. Since F is compact, we have $\|f\| \leqslant M$ for every $f \in F$.

We take a ball $B \equiv B(0, 2M) \subset C$.

Then

$$\inf_{c \in B} \sup_{f \in F} \|f - c\| = \inf_{c \in C} \sup_{f \in F} \|f - c\| \leqslant M.$$

The ball B is weakly compact in C and ϕ is a weakly lower semicontinuous function on B. Therefore ϕ attains its infimum in B for some $\mathbf{c} \in B$, say, which is a best simultaneous approximation to F, i.e.,

$$d(F, C) = \sup_{f \in F} \|f - \mathbf{c}\|.$$

Uniqueness follows as in Theorem 1.

The above theorem is given in [1]; we have proved Theorem 2 along the lines given in [5].

References

1. A. P. BOSZNAY, A remark on simultaneous approximation, *J. Appox. Theory* **23** (1978), 296–298.
2. J. B. DIAZ AND H. W. MCLAUGHLIN, Simultaneous approximation of a set of bounded real functions, *Math. Comp.* **23** (1969), 583–593.
3. J. B. DIAZ AND H. W. MCLAUGHLIN, On simultaneous Chebyshev approximation and Chebyshev approximation with an additive weight function, *J. Approx. Theory* **6** (1972), 68–71.
4. C. B. DUNHAM, Simultaneous Chebyshev approximation of functions on an interval, *Proc. Amer. Math. Soc.* **18** (1967), 472–477.
5. A. S. HOLLAND, B. N. SAHNEY, AND J. TZIMBALARIO, On best simultaneous approximation, *J. Indian Math. Soc.* **40** (1976), 69-73.